

# Exact solution of a one-dimensional Boltzmann equation for a granular tracer particle

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## Abstract

We consider a one-dimensional system consisting of a granular tracer particle of mass  $M$  in a bath of thermalized particles each of mass  $m$ . When the mass ratio,  $M/m$ , is equal to the coefficient of restitution,  $\alpha$ , the system maps to a one-dimensional elastic gas. In this case, Boltzmann equation can be solved exactly. We also obtain expressions for the velocity autocorrelation function and the diffusion coefficient. Numerical simulations of the Boltzmann equation are performed for  $M/m \neq \alpha$  where no analytical solution is available. It appears that the dynamical features remain qualitatively similar to those found in the exactly solvable case.

*Key words:* Boltzmann equation; granular gas; Green-Kubo relation.

*PACS:* 05.20.Dd, 45.70.-n,

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## 1 Introduction

Granular gases consist of particles that undergo dissipative collisions, making them a paradigm of non-equilibrium statistical mechanics [1, 2]. The absence of microscopic reversibility (in each collision, kinetic energy is lost) and the contraction of the phase space volume lead to specific properties which differ from elastically colliding thermal systems. These include non-gaussian statistics, modified hydrodynamics, and absence of equipartition [1, 2, 3, 4, 5, 6].

Compared to equilibrium, our knowledge of nonequilibrium statistical mechanics is still incomplete, and exact results are rare. In this paper, we consider a

one-dimensional model of a tracer particle that undergoes dissipative collisions with particles of a thermalized bath. The kinetic description is provided by the Boltzmann equation. We show that when the mass ratio of the tracer to a bath particle,  $M/m$ , is equal to the coefficient of restitution  $\alpha$ , there exists a mapping with the Boltzmann equation of identical elastic hard rods, the solution of which was derived almost thirty years ago [7]. We propose here a simpler way of solving the Boltzmann equation and we discuss the results in the context of our model, i.e. a granular particle in a thermalized bath. In addition, we obtain the velocity autocorrelation function as well as the diffusion coefficient. Numerical simulations of the Boltzmann equation show that the qualitative kinetic features are not strongly dependent on the value of the coefficient of restitution, implying that the exact solution provides the characteristics of the system also when  $M/m \neq \alpha$ , where an analytic solution is not available.

## 2 The model

We consider a one-dimensional gas of identical hard rods of mass  $m$  into which one inserts a granular hard rod of mass  $M$ [8]. Particles have hard core interactions and collisions between bath particles are elastic. But, collisions between the tracer particle and the bath particles are assumed inelastic and characterized by a coefficient of restitution,  $\alpha \leq 1$ . In addition, we assume that a stationary, Gaussian velocity distribution is imposed on the bath particles, compensating for the kinetic energy lost during collisions with the tracer particle.

For each collision, momentum is conserved so that for a collision between a bath particle and the tracer particle, one has

$$MV^* + mv^* = MV + mv \quad (1)$$

where the upper-case velocity corresponds to the tracer particle and the lower-case velocity to the bath particle, and the asterisks denote post-collisional quantities. At the moment of impact, the relative velocity changes sign and shrinks (for  $\alpha < 1$ ) according to the collision rule

$$V^* - v^* = -\alpha(V - v) \quad (2)$$

By combining Eqs.(1) and (2), the velocities of colliding particles after collision

are given by

$$V^* = \frac{(\mu - \alpha)V + (1 + \alpha)v}{1 + \mu} \quad (3)$$

$$v^* = \frac{(1 + \alpha)V + (\mu^{-1} - \alpha)v}{1 + \mu^{-1}} \quad (4)$$

where  $\mu = M/m$ .

Note that the tracer mass can be also taken as equal to the fluid particle mass by introducing an effective restitution coefficient[11].

The precollisional velocities  $V^{**}$  and  $v^{**}$ , corresponding to the inverse collision

$$(V^{**}, v^{**}) \rightarrow (V, v),$$

can be expressed in terms of  $V$  and  $v$  by replacing in Eqs.(3) and (4) the coefficient  $\alpha$  by  $\alpha^{-1}$ .

### 3 Structure of the Boltzmann equation at $\alpha = M/m$

The kinetic description of this model is provided by the Boltzmann equation according to which the probability distribution of the tracer particle evolves as

$$\begin{aligned} & \left( \frac{\partial}{\partial t} + V \frac{\partial}{\partial R} \right) f(R, V, t) = \\ & = \rho \int_{-\infty}^{+\infty} dc |V - c| \left[ \alpha^{-2} f(R, V^{**}, t) \phi(c^{**}) - f(R, V, t) \phi(c) \right] \end{aligned} \quad (5)$$

where  $R$  is the position of the tracer particle,  $\phi(c)$  denotes the equilibrium Maxwell distribution of the bath particles, and  $\rho$  their number density. Note that Eq. (5), although linear in the case of a tracer particle, has not yet been solved exactly. Exact solutions have, however, been found in the framework of the inelastic Maxwell model where the collision frequency, which occurs in the Boltzmann collisional term, is assumed independent of the relative velocity [9, 10].

The first objective of this paper is to show that an exact solution of Eq. (5) can be obtained when the coefficient of restitution equals the mass ratio of the tracer to a bath particle,  $\alpha = \mu = M/m$ . In this case, the Boltzmann equation for the distribution of the granular tracer particle turns out to be mathematically identical to the Boltzmann equation describing a tagged elastic particle

in a thermalized bath of mechanically identical particles. This fact, shown below, opens the way to an exact solution. However, the physical situations are totally different: evolution of a dissipative system towards a stationary state in one case and evolution of a conservative system towards thermal equilibrium in the other.

The differences between the two systems on the level of the microscopic dynamics are illustrated by the collision rules. For the elastic system with identical masses, a binary collision leads to the exchange of particle velocities,  $V^* = c$  and  $c^* = V$ . Moreover, for the inverse collision one finds  $V^{**} = c$  and  $c^{**} = V$ , which is related to the microscopic reversibility. For the granular tracer particle of mass  $M = \alpha m$  in a thermalized bath of particles of mass  $m$ , the collision rules are  $c^* = \alpha V + (1 - \alpha)c$  and  $V^* = c$ , whereas for the precollisional velocities corresponding to the inverse collision one finds  $c^{**} = V$  and  $V^{**} = c + (1 - \alpha)V$ . Despite the differences, in the gain term of the Boltzmann collision operator, in both cases, the precollisional velocity of the bath particle equals  $V$ , and is thus independent of  $c$ . And this common feature results in the simplification that allows the equation to be solved, as can be seen below.

For the sake of simplicity, we first consider the Boltzmann equation for spatially homogeneous systems. The general solution for inhomogeneous initial conditions can be derived in a very similar way. The corresponding, rather tedious, calculations are given in Appendix A.

The Boltzmann equation for homogeneous states takes the form

$$\begin{aligned} \frac{\partial f}{\partial t}(V, t) = & \rho \int_{-\infty}^{+\infty} dc |V - c| \left[ \alpha^{-2} f \left( \frac{(\mu - \alpha^{-1})V + (1 + \alpha^{-1})c}{1 + \mu}, t \right) \right. \\ & \times \phi \left( \frac{(1 + \alpha^{-1})V + (\mu^{-1} - \alpha^{-1})c}{1 + \mu^{-1}} \right) - f(V, t) \phi(c) \Big]. \end{aligned} \quad (6)$$

Putting  $\mu = \alpha$  in the gain term of Eq.(6) yields

$$\left( \frac{\partial}{\partial t} + \nu(V) \right) f(V, t) = \rho \phi(V) \int_{-\infty}^{+\infty} dc |V - c| \alpha^{-2} f \left( \frac{(\alpha - 1)V}{\alpha} + \frac{c}{\alpha}, t \right) \quad (7)$$

where  $\nu(V)$ , the collision rate of the tracer, is given by

$$\nu(V) = \rho \int_{-\infty}^{+\infty} dc |V - c| \phi(c) \quad (8)$$

By using a change of the integration variable,  $u = c/\alpha + (\alpha - 1)V/\alpha$ , in the

right-hand side of Eq.(7), one finally obtains

$$\left(\frac{\partial}{\partial t} + \nu(V)\right) f(V, t) = \rho \phi(V) \int_{-\infty}^{+\infty} du |V - u| f(u, t), \quad (9)$$

which coincides exactly with the Boltzmann equation for elastically colliding identical particles of mass  $m$ , solved by Résibois [7].

Before determining the time evolution of  $f(V, t)$ , we first focus on the stationary state  $f^{st}(V)$  which satisfies the equation obtained by dropping the time derivative in Eq.(9):

$$f^{st}(V) \int_{-\infty}^{+\infty} dc |V - c| \phi(c) = \phi(V) \int_{-\infty}^{+\infty} dc |V - c| f^{st}(c), \quad (10)$$

The solution of (10) is given by

$$f^{st}(V) = \phi(V) = \sqrt{\frac{m}{2\pi T}} \exp\left(-\frac{mV^2}{2T}\right) = \sqrt{\frac{M}{2\pi\alpha T}} \exp\left(-\frac{MV^2}{2\alpha T}\right) \quad (11)$$

where  $T$  is temperature of the bath. Therefore, the stationary velocity distribution is Maxwellian, but with a temperature  $\alpha T$  smaller than the bath temperature. This result is a specific case of a previously obtained general formula valid for any mass ratio in any dimension [1].

From Eq. (8) we get an explicit expression for the collision frequency of the granular particle

$$\nu(V) = \rho \left[ \sqrt{\frac{2\alpha T}{\pi M}} \exp\left(-\frac{MV^2}{2\alpha T}\right) + V \operatorname{erf}\left(\sqrt{\frac{M}{2\alpha T}} V\right) \right] \quad (12)$$

where  $\operatorname{erf}(x)$  is the error function.

In the following, we use dimensionless variables: the units of length and time are  $1/\rho$  and  $\sqrt{m}/\sqrt{T}\rho$ , respectively. Also, for the sake of simplicity, we use lower-case variables since no confusion with the variables associated with the bath particles is possible.

## 4 Exact Homogeneous solution

We first show that the integro-differential equation (9) can be replaced by a differential equation. To this end we introduce an auxiliary function  $G(u, t)$

$$G(u, t) = \int_{-\infty}^{+\infty} dc |u - c| f(c, t) \quad (13)$$

satisfying the relation

$$\frac{\partial^2 G}{\partial u^2}(u, t) = 2f(u, t). \quad (14)$$

Therefore, the Boltzmann equation, Eq.(9), when re-expressed in terms of  $G(u, t)$ , yields

$$\frac{\partial^3 G}{\partial t \partial u^2}(u, t) + \nu(u) \frac{\partial^2 G}{\partial u^2}(u, t) = \frac{d^2 \nu}{du^2}(u) G(u, t) \quad (15)$$

Let us introduce the time Laplace transform

$$\tilde{G}(u, z) = \int_0^{+\infty} dt G(u, t) e^{-zt}, \quad (16)$$

which when applied to Eq.(15) yields

$$[z + \nu(u)] \frac{\partial^2 \tilde{G}}{\partial u^2}(u, z) - \frac{d^2 \nu(u)}{du^2} \tilde{G}(u, z) = \frac{\partial^2 G}{\partial u^2}(u, 0) \quad (17)$$

By using the identity

$$\nu(u) \frac{\partial^2 \tilde{G}}{\partial u^2}(u, z) - \frac{d^2 \nu}{du^2}(u) \tilde{G}(u, z) = \frac{\partial}{\partial u} \left[ \nu(u) \frac{\partial \tilde{G}}{\partial u}(u, z) - \frac{d\nu}{du}(u) \tilde{G}(u, z) \right], \quad (18)$$

we can integrate Eq.(17) over the velocity from  $-\infty$  to  $u$  finding

$$[z + \nu(u)] \frac{\partial \tilde{G}}{\partial u}(u, z) - \frac{d\nu}{du}(u) \tilde{G}(u, z) = \frac{\partial G}{\partial u}(u, 0) + \tilde{B}(z) \quad (19)$$

where  $\tilde{B}(z)$  is a function that can be determined by taking in (19) the limit  $u \rightarrow \infty$ . According to the definitions (13),(16) in the region of large velocities  $\tilde{G}(u, z)$  is asymptotically given by

$$\tilde{G}(u, z) = \frac{|u|}{z} - \text{sgn}(u) < \tilde{v}(z) > \quad (20)$$

where  $< \tilde{v}(z) >$  is the Laplace transform of the mean velocity

$$< \tilde{v}(z) > = \int_{-\infty}^{+\infty} du u \tilde{f}(u, z). \quad (21)$$

Using Eq.(20) one can readily check that Eq.(19) considered in the limit  $|u| \rightarrow \infty$  reduces to the equality  $\tilde{B}(z) = < \tilde{v}(z) >$ .

Introducing now the function

$$\tilde{H}(u, z) = \frac{\tilde{G}(u, z)}{[z + \nu(u)]} \quad (22)$$

we rewrite Eq.(19) as

$$\frac{\partial \tilde{H}}{\partial u}(u, z) = \frac{1}{[z + \nu(u)]^2} \left[ \frac{\partial G}{\partial u}(u, 0) + \langle \tilde{v}(z) \rangle \right] \quad (23)$$

From the asymptotic formula Eq.(20) combined with the definition (22), it follows that

$$\lim_{u \rightarrow \pm\infty} \tilde{H}(u, z) = \frac{1}{z}. \quad (24)$$

Using this result we can integrate Eq.(23) over the velocity space obtaining an explicit expression for the mean velocity

$$\langle \tilde{v}(z) \rangle = - \int_{-\infty}^{+\infty} \frac{du}{[z + \nu(u)]^2} \frac{\partial G}{\partial u}(u, 0) \left( \int_{-\infty}^{+\infty} \frac{du}{[z + \nu(u)]^2} \right)^{-1} \quad (25)$$

The initial condition appears in Eq. (25) through the derivative (see Eq. (13))

$$\frac{\partial G}{\partial u}(u, 0) = \int_{-\infty}^u dc f(c, 0) - \int_u^{+\infty} dc f(c, 0) \quad (26)$$

It is worth noting that we could calculate the first moment of the probability distribution  $\langle \tilde{v}(z) \rangle$  without yet knowing the complete solution of the Boltzmann equation. If  $f(u, 0)$  is an even function of velocity, then the derivative (Eq. 26) is an odd function, and from Eq. (25) we get  $\langle \tilde{v}(z) \rangle = 0$ , and hence the mean velocity  $\langle v(t) \rangle = 0$ , for any  $t > 0$ .

The differential equation, Eq.(23) can be easily integrated with the use of the boundary condition, Eq.(24). The relation Eq. (22) between  $\tilde{G}(u, z)$  and  $\tilde{H}(u, z)$  yields then the complete solution of Eq.(19)

$$\tilde{G}(u, z) = [z + \nu(u)] \left[ \int_{-\infty}^u dw \frac{1}{[z + \nu(w)]^2} \left( \frac{\partial G}{\partial w}(w, 0) + \langle \tilde{v}(z) \rangle \right) + \frac{1}{z} \right] \quad (27)$$

Differentiating Eq.(27) twice with respect to  $u$  we find the Laplace transform  $\tilde{f}(u, z)$  of the solution of the Boltzmann equation

$$\begin{aligned} \tilde{f}(u, z) = & \phi(u) \left[ \frac{1}{z} + \int_{-\infty}^u dw \frac{1}{[z + \nu(w)]^2} \left( \frac{\partial G}{\partial w}(w, 0) + \langle \tilde{v}(z) \rangle \right) \right] \\ & + \frac{1}{[z + \nu(u)]} f(u, 0). \end{aligned} \quad (28)$$

Eq.(28) when integrated over the velocity yields  $\int dv \tilde{f}(u, z) = 1/z$ , a result consistent with the conservation of the probability.

The use of the identity

$$\frac{1}{[z + \nu(u)]} + \int_{-\infty}^u dw \frac{1}{[z + \nu(w)]^2} \frac{d\nu(w)}{dw} = 0$$

in Eq.(28) allows us to perform the inverse Laplace transform term by term. In this way we determine the structure of the time dependent solution of the Boltzmann equation

$$f(u, t) = \phi(u) + [f(u, 0) - \phi(u)]e^{-t\nu(u)} + \phi(u) \left[ \int_{-\infty}^u dw \left( \left[ \frac{\partial G}{\partial w}(w, 0) - \frac{d\nu(w)}{dw} \right] te^{-t\nu(w)} + A(w, t) \right) \right] \quad (29)$$

where  $A(w, t)$  is the inverse Laplace transform of

$$\tilde{A}(w, z) = \frac{\langle \tilde{v}(z) \rangle}{[z + \nu(w)]^2}. \quad (30)$$

Note that when the initial distribution  $f(u, 0)$  is an even function of  $u$  (and thus  $\langle \tilde{v}(z) \rangle = 0$ ),  $A(w, t)$  vanishes.

The first term on the right-hand side of Eq.(29) represents the stationary solution, whereas the second term describes the exponential decay (with relaxation time  $1/\nu(u)$ ) of the initial deviation from the stationary distribution. For high velocities the collision frequency  $\nu(u)$  goes as  $|u|$  so that the associated relaxation time approaches 0 as  $|u|^{-1}$ . Consequently, the high energy tails of the distribution function converge faster towards the stationary state than the rest of the distribution. This phenomenon is illustrated in Fig.4 which shows the evolution of the velocity distribution starting from a “gate” function. The initial discontinuities in velocity space are present at any finite time. Their position remains fixed, but their magnitude decreases with time. Finally, we note that the last term on the right-hand side of Eq.(29) vanishes both for  $t \rightarrow 0$ , and for  $t \rightarrow \infty$ .

## 5 Velocity autocorrelation function and diffusion coefficient

The velocity autocorrelation function is defined by

$$C(t) = \langle v(t)v(0) \rangle \quad (31)$$



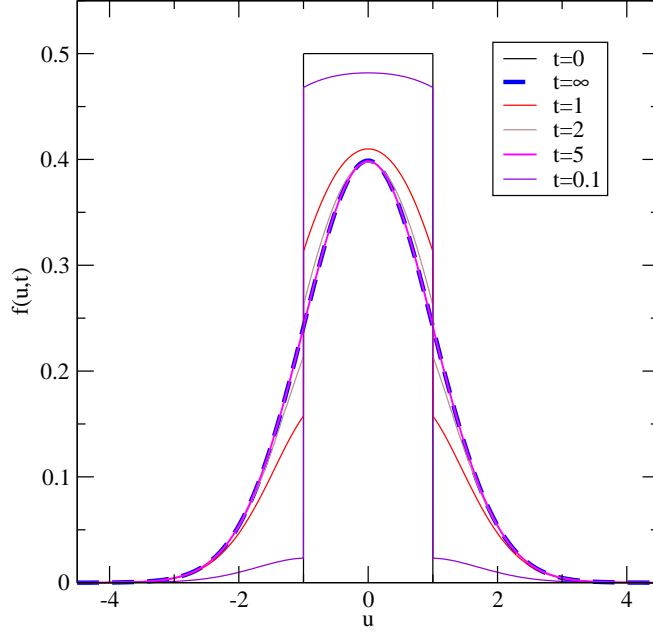


Fig. 1. Velocity distribution function  $f(u, t)$  versus  $u$  at different times  $t = 0.1, 1, 2, 5, \infty$ .

where the brackets denote an average over the stationary Maxwell distribution (Eq. 11). We can calculate  $C(t)$  exactly using the solution (28). Specifically,

$$C(t) = \int_{-\infty}^{+\infty} v F(v, t) dv \quad (32)$$

where  $F(v, t)$  is the solution of the Boltzmann equation corresponding to the initial condition

$$F(v, 0) = v \phi(v) \quad (33)$$

The Laplace transform of  $C(t)$

$$\tilde{C}(z) = \int_{-\infty}^{+\infty} v \tilde{F}(v, z) dv = \langle \tilde{V}(z) \rangle \quad (34)$$

is thus equal to the Laplace transform of the mean velocity  $\langle V(t) \rangle$  corresponding to the solution  $\tilde{F}(v, z)$  (see Eq.(25)). Note that the initial condition (33) does not represent a probability distribution. One finds in this specific case the relation

$$\frac{\partial G}{\partial u}(u, 0) = u \frac{d\nu(u)}{du} - \nu(u). \quad (35)$$

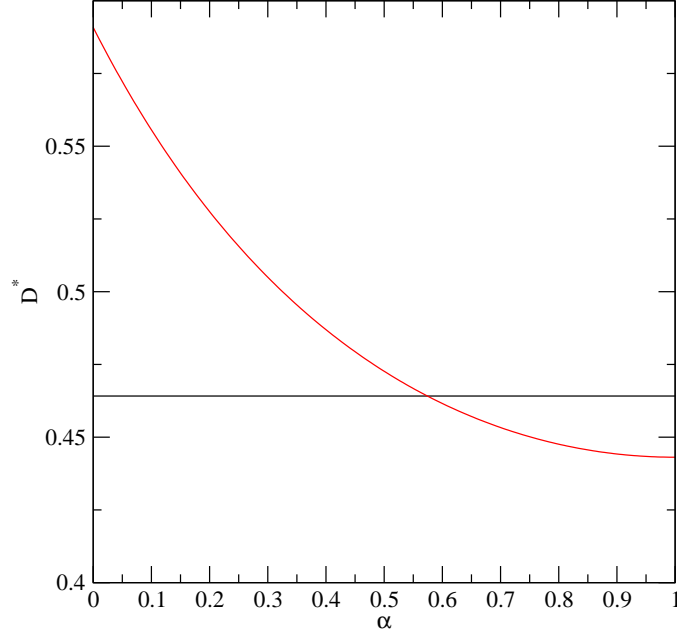


Fig. 2. Reduced diffusion coefficient as a function of the coefficient of restitution: the horizontal line corresponds to the exact solution of the Boltzmann equation and the curve to the first Sonine approximation.

Inserting Eq.(35) into Eq.(25) after integration by parts we get

$$\tilde{C}(z) = \langle \tilde{V}(z) \rangle = 2 \left( \int_{-\infty}^{+\infty} \frac{du}{[z + \nu(u)]^2} \right)^{-1} - z. \quad (36)$$

Eq.(36) is an exact formula for the velocity autocorrelation function. The above derivation is by far simpler than that given in [7].

The dimensionless diffusion coefficient  $D^* = \tilde{C}(0)$  can be now deduced by taking the limit  $z \rightarrow 0$  in Eq.(36). We find

$$D^* = \left[ \int_0^{+\infty} \frac{du}{\nu(u)^2} \right]^{-1} \quad (37)$$

Restoring the physical dimensions of the variables yields for the diffusion coefficient  $D$  the formula

$$D = \frac{1}{\rho} \sqrt{\frac{\alpha T}{M}} D^* \quad (38)$$

The numerical value of  $D^*$  is 0.464139.... It is worth noting that  $D$  is propor-

tional to the square root of the coefficient of restitution and hence vanishes when  $\alpha \rightarrow 0$ .

The diffusion coefficient for a tracer particle has also been obtained approximately from a normal Chapman-Enskog solution of the Boltzmann equation [12, 13] yielding

$$D^* = \frac{\sqrt{\pi}}{3 + 2\alpha - \alpha^2}. \quad (39)$$

Figure 2 shows the reduced (dimensionless) diffusion coefficient  $D^*$  versus the coefficient of restitution  $\alpha$ .

We note that the diffusion coefficient for elastic ( $\alpha = 1$ ) hard rods in one-dimension is known exactly:  $D^* = 1/\sqrt{2\pi} \simeq 0.3989$ ..[14]. Probably fortuitously, at  $\alpha = 1$  the lowest order Sonine approximation to the solution of the Boltzmann equation gives a result for  $D^*$  that is closer to this exact result than the prediction of the Boltzmann equation itself. When  $\alpha < 1$ , the reduced diffusion coefficient within the Sonine approximation increases when  $\alpha$  decreases, whereas the result (Eq. 37) provided by the Boltzmann equation is independent of  $\alpha$ . The two curves cross at  $\alpha = 0.574$ . In both cases, of course, the full diffusion coefficient  $D$  vanishes when  $\alpha$  goes to zero due to the square-root dependence on the coefficient of restitution in Eq.(38).

## 6 Simulation results

In order to study the situations where  $M/m \neq \alpha$ , we have solved the Boltzmann equation numerically by using a DSMC method [15]. We focus here on the stationary dynamics. Figure 3 displays both the numerical results of the simulation and the exact results obtained by an inverse Laplace transform of Eq.(36): the ratio of the tracer particle over the mass of the bath particle is equal to  $\alpha$ , when  $\alpha = 0.6$ . Note that decreasing the coefficient of restitution increases the relaxation time of the normalized velocity autocorrelation function but does not significantly change its shape. The dashed curve corresponds to the exact solution, Eq. (36), and matches very accurately the simulation results when  $\alpha = 0.6$ .

As  $\alpha$  decreases, the velocity autocorrelation function decreases more slowly. This behavior is, at first glance, counterintuitive. When  $\alpha$  is small the postcollisional velocity of the tracer is smaller than when  $\alpha$  is larger. One therefore expects that the correlation function should approach zero more rapidly when  $\alpha$  is small. This reasoning, however, does not account for the time between collisions which increases as  $\alpha$  decreases. Evidently it is this effect that dominates, leading to the observed behavior.

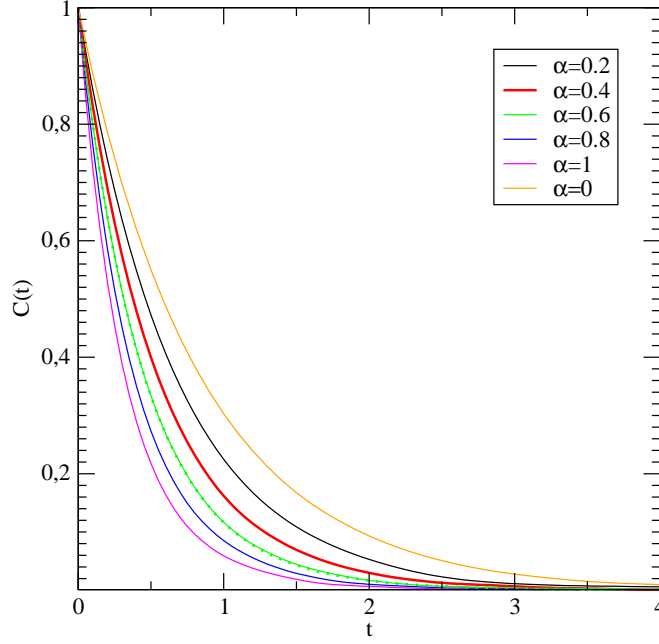


Fig. 3. Normalized autocorrelation function  $C(t)$  of the velocity versus time for  $M = 0.6m$  and for various values of the coefficient of restitution  $\alpha = 1.0, 0.8, 0.6, 0.4, 0.2, 0$ . The dashed curve corresponds to the inverse Laplace transform of Eq.(36)

## 7 Self correlation function

The more general case of spatially inhomogeneous states also admits an analytic solution of the Boltzmann equation. Since the strategy is basically similar to that developed in Section 3, details of calculations are given in Appendix A. The final result for  $\tilde{f}(q, u, z)$  reads

$$\begin{aligned} \tilde{f}(q, u, z) = & \phi(u) \left[ \frac{1}{z} - \frac{1}{z + \nu(u) + iqu} + \right. \\ & \left. \int_{-\infty}^u dw \frac{1}{[z + \nu(w) + iqw]^2} \left( \frac{\partial \hat{G}}{\partial w}(q, w, 0) - \frac{d\nu}{dw}(w) + \langle \tilde{v}(q, z) \rangle \right) \right] \\ & + \frac{1}{z + \nu(u) + iqu} f(q, u, 0). \end{aligned} \quad (40)$$

Here  $\hat{G}$  denotes the Fourier transform

$$\hat{G}(q, u, 0) = \int_{-\infty}^{+\infty} dr e^{iqr} G(r, u, 0) \quad (41)$$

and  $G(r, u, t)$  is defined by Eq.(A.1) in analogy with the definition (13).

The conditional density  $F(r, t)$  can be obtained from the inhomogeneous Boltzmann equation by integrating  $f(r, v, t)$  over the velocities assuming that

$$f(r, v, 0) = \delta(r)\phi_M(u) \quad (42)$$

i.e. that initially the tracer particle was located at the origin with the equilibrium velocity distribution. This is in fact the definition of the self-correlation function

$$F(r, t) = \int_{-\infty}^{+\infty} dv f(r, v, t). \quad (43)$$

As, in accordance with the definition (A.1),

$$2f(r, v, t) = \frac{\partial^2}{\partial v^2} G(r, v, t) \quad (44)$$

one finds that the Fourier-Laplace transform of  $F(r, t)$  is given by

$$\tilde{F}(q, z) = \int dr e^{-iqr} \int_0^\infty dt e^{-zt} F(r, t) = \frac{1}{2} \left[ \frac{\partial \tilde{G}}{\partial v}(q, +\infty, z) - \frac{\partial \tilde{G}}{\partial v}(q, -\infty, z) \right] \quad (45)$$

From Eq.(A.13) a simple expression follows

$$\tilde{F}(q, z) = \frac{1 - iq \langle \tilde{v}(q, z) \rangle}{z}, \quad (46)$$

whereas using the initial condition Eq.(42) one finds the formula

$$\langle \tilde{v}(q, z) \rangle = iq \left[ \frac{z(1 + q^2)J(q, z) - 2}{z(1 + q^2)J(q, z) + 2q^2} \right] \quad (47)$$

with

$$\tilde{J}(q, z) = \int_{-\infty}^{+\infty} \frac{dw}{[z + iqw + \nu(w)]^2} \quad (48)$$

Finally, one gets

$$\tilde{F}(q, z) = \frac{1}{z} + \frac{q^2}{z} \left( 1 - \frac{2(1 + q^2)}{z(1 + q^2)\tilde{J}(q, z) + 2q^2} \right) \quad (49)$$

The wave vector dependent poles of  $\tilde{F}(q, z)$  satisfy the equation

$$z(1 + q^2)\tilde{J}(q, z) + 2q^2 = 0 \quad (50)$$

For the hydrodynamic pole  $z(q)$ , which tends to zero when  $q \rightarrow 0$ , Eq.(50) to dominant order yields

$$z\tilde{J}(0, 0) + 2q^2 = 0 \quad (51)$$

$$z + Dq^2 = 0 \quad (52)$$

where  $D$  is the diffusion coefficient derived in section 5 with the use of the Green-Kubo autocorrelation formula (see Eq.(37) ). Here it appears in the hydrodynamic long-time and long-length scale limit of the solution of the Boltzmann equation which expresses the density fluctuation of the tracer particle.

## 8 Conclusion

We have shown that, within the framework of the one-dimensional Boltzmann equation, the dynamics of a granular tracer particle can be mapped onto the dynamics of an elastic tracer particle with a renormalized mass. A recent article of Santos and Dufty [16] shows that this result can be extended to more general situations.

J. P. acknowledges financial support by the CNRS, France, and the hospitality at the *Laboratoire de Physique Théorique de la Matière Condensée, UPMC (Paris)* where this research has been carried out.

## A Inhomogeneous solutions of the Boltzmann equation

We summarize the basic steps leading to the solution of the inhomogeneous Boltzmann equation. The quantity which generalizes Eq.(13) is the function  $G(r, u, t)$  defined as

$$G(r, u, t) = \int_{-\infty}^{+\infty} dc |u - c| f(r, c, t). \quad (\text{A.1})$$

Let us denote  $\tilde{G}(q, u, z)$  the Fourier-Laplace transform of  $G(r, u, t)$ ,

$$\tilde{G}(q, u, z) = \int_{-\infty}^{+\infty} dr e^{iqr} \int_0^{+\infty} dt e^{-zt} G(r, u, t). \quad (\text{A.2})$$

The Boltzmann equation, Eq.(5), can be then rewritten as

$$[z + iqu + \nu(u)] \frac{\partial^2 \tilde{G}}{\partial u^2}(q, u, z) - \frac{d^2 \nu(u)}{du^2} \tilde{G}(q, u, z) = \frac{\partial^2 \hat{G}}{\partial u^2}(q, u, 0) \quad (\text{A.3})$$

Integrating Eq.(A.3) over the velocity interval  $] -\infty, u]$  yields

$$[z + iqu + \nu(u)] \frac{\partial \tilde{G}}{\partial u}(q, u, z) - \left[ \frac{\partial \nu(u)}{\partial u} + iq \right] \tilde{G}(q, u, z) = \frac{\partial \hat{G}}{\partial u}(q, u, 0) + \tilde{B}(q, z) \quad (\text{A.4})$$

where the function  $\tilde{B}(q, z)$  is to be determined. Similarly to Eq.(22), it is convenient to introduce the function  $\tilde{H}(q, u, z)$

$$\tilde{H}(q, u, z) = \frac{\tilde{G}(q, u, z)}{[z + iqu + \nu(u)]}, \quad (\text{A.5})$$

which satisfies a differential equation analogous to Eq.(23) with  $\nu(u)$  replaced by  $[\nu(u) + iqu]$ , and  $\langle \tilde{v}(z) \rangle$  replaced by  $\langle \tilde{v}(q, z) \rangle$ . Therefore, the solution of Eq.(A.3) can be expressed as

$$\begin{aligned} \tilde{G}(q, u, z) = & [z + \nu(u) + iqu] \\ & \times \left[ \int_{-\infty}^u dw \frac{1}{[z + \nu(w) + iqw]^2} \left( \frac{\partial \hat{G}}{\partial w}(q, w, 0) + \tilde{B}(q, z) \right) + D(q, z) \right] \end{aligned} \quad (\text{A.6})$$

The functions  $B(q, z)$  and  $D(q, z)$  can be determined from the boundary conditions satisfied by  $\tilde{G}(q, u, z)$  and  $\tilde{H}(q, u, z)$ . The asymptotic behavior of  $\tilde{G}(q, u, z)$  for large  $u$  at fixed  $z$  and  $q$  is given by

$$\tilde{G}(q, u, z) \sim |u| \int_{-\infty}^{+\infty} dw \tilde{f}(q, w, z) - \text{sgn}(u) \langle \tilde{v}(q, z) \rangle \quad (\text{A.7})$$

where

$$\langle \tilde{v}(q, z) \rangle = \int_{-\infty}^{+\infty} dw w \tilde{f}(q, w, z) \quad (\text{A.8})$$

The continuity equation (conservation of the probability) is expressed as

$$z \int_{-\infty}^{+\infty} dw \tilde{f}(q, w, z) + iq \langle \tilde{v}(q, z) \rangle = 1 \quad (\text{A.9})$$

By combining Eqs.(A.7) and (A.9), one obtains the asymptotic relation

$$\tilde{G}(q, u, z) \sim \frac{|u|}{z} [1 - iq \langle \tilde{v}(q, z) \rangle] - \text{sgn}(u) \langle \tilde{v}(q, z) \rangle \quad (\text{A.10})$$

The limit of  $\tilde{H}(q, u, z)$  when  $u \rightarrow \pm\infty$  is then given by

$$\lim_{u \rightarrow \pm\infty} \tilde{H}(q, u, z) = \frac{1 - iq \langle \tilde{v}(q, z) \rangle}{(1 \pm iq)z} \quad (\text{A.11})$$

By integrating the differential equation obeyed by  $\tilde{H}(q, u, z)$  over velocity, one obtains a sum rule analogous to Eq.(25):

$$\begin{aligned} \langle \tilde{v}(q, z) \rangle = & - \left( \frac{2iq}{1 + q^2} + \int_{-\infty}^{+\infty} \frac{du}{[z + \nu(u) + iqu]^2} \frac{\partial \hat{G}}{\partial u}(q, u, 0) \right) \\ & \times \left( \int_{-\infty}^{+\infty} \frac{du}{[z + \nu(u) + iqu]^2} + \frac{2q^2}{1 + q^2} \right)^{-1} \end{aligned} \quad (\text{A.12})$$

The knowledge of the boundary conditions (A.10) and (A.11) permits one to write the solution (A.6) in an explicit form

$$\tilde{G}(q, u, z) = [z + \nu(u) + iqu] \quad (\text{A.13})$$

$$\times \left[ \int_{-\infty}^u dw \frac{1}{[z + \nu(w) + iqw]^2} \left( \frac{\partial \hat{G}}{\partial w}(q, w, 0) + \langle \tilde{v}(q, z) \rangle \right) + \frac{1 - iq \langle \tilde{v}(q, z) \rangle}{(1 - iq)z} \right]$$

Taking the second derivative of Eq.(A.13) with respect to the velocity variable  $u$  leads to Eq.(40).

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